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On Reverse Laplacian Energy of a Graph

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Abstract: Let G be a simple undirected graph with n vertices and m edges. The Laplacian matrix $L(G)$ of graph G is defined as $L(G) = (l_{ij})$, where (l_{ij}) is equal to -1 if v_i and v_j are adjacent, 1 if v_i and v_j are not adjacent and $d(v_i)$ if $i = j$,where $d(v_i)$ is the vertex degree of v_i. This paper defines the reverse Laplacian matrix, reverse Laplacian energy and find the same for some standard graphs. Further, we calculate upper for reverse Laplacian energy.

Keywords: reverse vertex degree; reverse Laplacian matrix; reverse Laplacian energy.

1. Introduction

Let G be a simple graph with n vertices and m size. Let $\Delta(G)$ denote the maximum degree among the vertices of G. The reverse vertex degree of a vertex v_i in G is defined as, $c_{v_i} = \Delta(G) - d(v_i) + 1$, where $d(v_i)$ is the degree of vertex v_i. Let $A = (a_{ij})$ be the adjacency matrix of the graph G. In 1978 [1] I. Gutman introduced the concept of energy of a graph. If $\mu_1, \mu_2, \dots, \mu_n$ are the eigenvalues of A, assumed in non-increasing order. Then, the energy $E(G)$ of the graph G is defined to be the sum of the absolute values of the eigenvalues of$G$. That is, $E(G) = \sum_{i=1}^{n} |\mu_i|$.

In 2006,[2] I. Gutman and B. Zhou defined the Laplacian energy of a graph G. Let G be a graph with n vertices and m size. The Laplacian matrix of the graph G, denoted by $L = (l_{ij})$, is a $n \times n$ matrix whose elements are defined as, $l_{ij} = -1$ if $v_i\ and\ v_j are\ adjacent$, 1 if $v_i and\ \ v_j are\ not\ adjacent$, $d(v_i)$ if $i = j$. If $\mu_1, \mu_2, \dots, \mu_n$ are the eigenvalues of $L(G)$ then, Laplacian energy $LE(G)$ is defined as $LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$.

The basic properties, upper and lower bounds for Laplacian energy, have been found in the literature[3-8]. Motivated by the definition of Laplacian matrix and reverse vertex degree, we define reverse Laplacian matrix and reverse Laplacian energy of a graph. We found reverse Laplacian energy for some standard graphs and further found bounds for the graph G.

Let G be a graph with n vertices and m size. Then, the reverse Laplacian matrix $L_R(G)$ is defined as $L_R(G) = (l_{r_{ij}})$ where, $l_{r_{ij}} = -1$ if $v_i\ and\ v_j are\ adjacent$, 1 if $v_i and\ \ v_j are\ not\ adjacent$, $c(v_i)$ i $i = j$. If $\mu_1, \mu_2 \dots \mu_n$ are the eigenvalues of $L_R(G)$ then, reverse Laplacian energy $EL_R(G)$ is defined as $EL_R(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$.

2. Materials and Methods

In literature, the reverse vertex degree of a vertex v_i in G is defined as, $c_{v_i} = \Delta(G) - d(v_i) + 1$, where $d(v_i)$ is a degree of the vertex v_i[9] and found many results in [12-13]. The energy of a graph plays an important role in chemistry. Based on this, several authors have worked on computing the polynomials of various chemical graphs and nanostructures, thereby finding their respective indices[14-15].

Likewise, we defined the new energy of the graph called reverse Laplacian energy and found some results.

3. Results and Discussion

3.1. Reverse Laplacian energy.

3.1.1. Theorem 1.

Let G be a simple graph with n vertices and m edges. If $\mu_1, \mu_2 \dots \mu_n$ are the eigenvalues of the Reverse Laplacian matrix $L_R(G)$, then

$$\text{i. } \sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} c_{v_i}$$

$$\text{ii. } \sum_{i=1}^{n} \mu_i^2 = 2|E| + \sum_{i=1}^{n} c_{v_i}^2$$

Proof:

(i) Since the sum of eigenvalues of the matrix $L_R(G)$ is the trace of the matrix $L_R(G)$. But from the definition, the principle diagonal elements are reverse vertex degree c_{v_i}, $i = 1,2,3, \dots n$. So we have,$\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} l_{r_{ii}} = \sum_{i=1}^{n} c_{v_i}$

(ii) Similar to above, the sum of the square of the eigenvalues of the matrix $L_R(G)$ is same as trace of $[L_R(G)]^2$. Therefore we have,

$$\begin{aligned}\sum_{i=1}^{n} \mu_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} l_{r_{ij}} l_{r_{ji}} \\ &= \sum_{i \neq j} l_{r_{ij}} l_{r_{ji}} + \sum_{i=1}^{n} l_{r_{ii}}^2 \\ &= 2 \sum_{1 \le i < j \le n} l_{r_{ij}}^2 + \sum_{i=1}^{n} l_{r_{ii}}^2 \\ &= 2|E| + \sum_{i=1}^{n} c_{v_i}^2\end{aligned}$$

3.1.2.Theorem 2.

For $n \geq 3$, the reverse Laplacian energy of the star graph S_n is $EL_R(S_n) = n^3 - 4n^2 + 8n - 4$

Proof:

Consider the Star graph S_n with vertex $v_1, v_2, \dots v_n$. Then, the maximum reverse degree matrix of the graph S_n

$$L_R(S_n) = \begin{bmatrix} 1 & -1 & -1 & \dots & -1 \\ -1 & n-1 & 0 & \dots & 0 \\ -1 & 0 & n-1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & n-1 \end{bmatrix}$$

Characteristic polynomial is

$$\begin{vmatrix} 1-\mu & -1 & -1 & \dots & -1 \\ -1 & n-1-\mu & 0 & \dots & 0 \\ -1 & 0 & n-1-\mu & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & n-1-\mu \end{vmatrix}$$

Characteristic equation is $(\mu)(\mu-(n-1))^{n-2}(\mu-n)=0$. The eigenvalues of reverse Laplacian matrix $L_R(G)$ of the graph are:

$\mu = 0$ (one times), $\mu = (n-1)$((n-2) times)), $\mu = n$(one time)

Number of vertices=n, Number of edges=$n-1$

∴ Average degree=$\frac{2(n-1)}{n}$

Hence the reverse Laplacian energy is,

$$\begin{aligned} EL_R(S_n) &= \left|0-\frac{2(n-1)}{n}\right|(1)+\left|(n-1)-\frac{2(n-1)}{n}\right|(n-2)+\left|n-\frac{2(n-1)}{n}\right|(1) \\ &= \frac{2(n-1)}{n}+\frac{n^2-3n+2}{n}(n-2)+\frac{n^2-2n_2}{n} \\ &= n^3-4n^2+8n-4 \end{aligned}$$

3.1.3.Theorem 3.

For $n \geq 3$, the reverse Laplacian energy of the Complete graph K_n is $EL_R(S_n) = n^2-2n$

Proof:

Consider the Complete graph K_n with vertex $v_1, v_2, \dots v_n$. Then, the reverse Laplacian matrix of the graph K_n

$$M_R(K_n) = \begin{bmatrix} 1 & -1 & -1 & \dots & -1 \\ -1 & 1 & -1 & \dots & -1 \\ -1 & -1 & 1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & 1 \end{bmatrix}$$

The characteristic polynomial is

$$\begin{vmatrix} 1-\mu & -1 & -1 & \dots & -1 \\ -1 & 1-\mu & -1 & \dots & -1 \\ -1 & -1 & 1-\mu & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & 1-\mu \end{vmatrix}$$

The characteristic equation is $(\mu-2)^{n-1}(\mu-(n-2))=0$. The eigenvalues of the maximum reverse degree matrix $L_R(G)$ of the graph are:

$\mu = 2$ $(n-1\ times)$, $\mu = -(n-2)$(one time)

Number of vertices=n, Number of edges=nC_2=$\frac{n(n-1)}{2}$

∴ Average degree=$\frac{2m}{n}$=$\frac{2\frac{n(n-2)}{2}}{n}$=$n-1$

Hence the reverse Laplacian energy is,

$$\begin{aligned} EL_R(K_n) &= |2-(n-1)|(n-1)+|-(n-2)-(n-1)| \\ &= [n-3](n-1)+2n-3 \\ &= n^2-2n \end{aligned}$$

3.1.4. Theorem 4.

For $n \geq 3$, the reverse Laplacian energy of the complete bipartite graph $K_{n,n}$ is $EL_R(K_{n,n}) = 2n^2 - 2n + 2$

Proof:

Consider the complete bipartite graph $K_{n,n}$ with vertex $v_1, v_2, \dots v_n$. Then, the reverse Laplacian matrix of the graph $K_{n,n}$ is given by,

$$\mathrm{M}_R(K_{n,n}) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & -1 & -1 & -1 & \dots & -1 \\ 0 & 1 & 0 & \dots & 0 & -1 & -1 & -1 & \dots & -1 \\ 0 & 0 & 1 & \dots & 0 & -1 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 & 1 & 0 & 0 & \dots & 0 \\ -1 & -1 & -1 & \dots & -1 & 0 & 1 & 0 & \dots & 0 \\ -1 & -1 & -1 & \dots & -1 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & -1 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{(2n\times 2n)}$$

The characteristic polynomial is

$$\begin{vmatrix} 1-\mu & 0 & 0 & \dots & 0 & -1 & -1 & -1 & \dots & -1 \\ 0 & 1-\mu & 0 & \dots & 0 & -1 & -1 & -1 & \dots & -1 \\ 0 & 0 & 1-\mu & \dots & 0 & -1 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1-\mu & -1 & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 & 1-\mu & 0 & 0 & \dots & 0 \\ -1 & -1 & -1 & \dots & -1 & 0 & 1-\mu & 0 & \dots & 0 \\ -1 & -1 & -1 & \dots & -1 & 0 & 0 & 1-\mu & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & -1 & 0 & 0 & 0 & \dots & 1-\mu \end{vmatrix}$$

The characteristic equation is $(\mu-1)^{2n-2}(\mu-(n+1))(\mu+(n-1)) = 0$
The eigenvalues of the maximum reverse degree matrix $M_R(K_{n,n})$ of the graph are:

$\mu = 1$ (2n-2 times), $\mu = (n+1)$(one time), $\mu = -(n-1)$ (one time)

Number of vertices=$2n$, Number of edges=n^2

$\therefore$ Average degree=$\frac{2m}{n}$=$\frac{2n^2}{2n}$=n

Hence the reverse Laplacian energy is,

$$\begin{aligned} EL_R(K_{n,n}) &= |1-(n)|(2n-2)+|(n+1)-(n)|(1)+|-(n-1)-(n)| \\ &= 2n^2-2n+2 \end{aligned}$$

Theorem 3.1.5.

For $n \geq 3$, the reverse Laplacian energy of the Crown graph S_n^0 is $EL_R(S_n^0) = 2n^2 - 4n + 2$

Proof:

Consider the crown graph S_n^0 with vertex $v_1, v_2, \dots v_n$. Then, the reverse Laplacian matrix of the graph S_n^0

$$M_R(S_n^0) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & -1 & -1 & \dots & -1 \\ 0 & 1 & 0 & \dots & 0 & -1 & 0 & -1 & \dots & -1 \\ 0 & 0 & 1 & \dots & 0 & -1 & -1 & 0 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 & -1 & -1 & \dots & 0 \\ 0 & -1 & -1 & \dots & -1 & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & -1 & \dots & -1 & 0 & 1 & 0 & \dots & 0 \\ -1 & -1 & 0 & \dots & -1 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{(2n\times 2n)}$$

Characteristic polynomial is

$$\begin{vmatrix} 1-\mu & 0 & 0 & \dots & 0 & 0 & -1 & -1 & \dots & -1 \\ 0 & 1-\mu & 0 & \dots & 0 & -1 & 0 & -1 & \dots & -1 \\ 0 & 0 & 1-\mu & \dots & 0 & -1 & -1 & 0 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1-\mu & -1 & -1 & -1 & \dots & 0 \\ 0 & -1 & -1 & \dots & -1 & 1-\mu & 0 & 0 & \dots & 0 \\ -1 & 0 & -1 & \dots & -1 & 0 & 1-\mu & 0 & \dots & 0 \\ -1 & -1 & 0 & \dots & -1 & 0 & 0 & 1-\mu & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & 0 & 0 & 0 & 0 & \dots & 1-\mu \end{vmatrix}$$

The characteristic equation is $\mu^{(n-1)}(\mu-2)^{(n-1)}(\mu+(n-2))(\mu-n)=0$
The eigenvalues of reverse Laplacian matrix $M_R(S_n^0)$ of the graph are:

$\mu = 0$ ((n-1) times), $\mu = 2$ ((n-1) times), $\mu = -(n-2)$(one time), $\mu = n$(one time)

Number of vertices=$2n$, Number of edges=$n(n-1)$

$\therefore$ Average degree=$\frac{2m}{n}$=$\frac{2n(n-1)}{2n}$=$n-1$

Hence the reverse Laplacian energy is,

$$\begin{aligned} EL_R(S_n^0) &= |0-(n-1)|(n-1)+|2-(n-1)|(n-1)+|-(n-2)-(n-1)|(1)+|n-(n-1)|(1) \\ &= (n-1)^2+(n-3)(n-1)+(2n-3)+1 \\ &= 2n^2-4n+2 \end{aligned}$$

3.2. Bounds for the reverse Laplacian energy.

In this section, lower and upper bounds for the reverse Laplacian energy of graphs are calculated.To do this, we need the following known inequality.

3.2.1 Lamma (Cauchy–Schwarz inequality) .

For all sequences of real number a_i and b_i, $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$, equality holds if and only if $a_i = kb_i$ for a non zero constant $k \in \mathbf{R}$

3.2.2. Theorem (upper bound).

Let G be a simple graph with n vertices, m size. Let $\mu_1, \mu_2, \dots \mu_n$ be the eigenvalues of the reverse Laplacian matrix $L_R(G)$ then, $EL_R(G) \leq \sqrt{n\left(2|E| + \sum_{i=1}^{n} c_{v_i}^2\right) + 2|E|}$

Proof:

Let $\mu_1, \mu_2, \dots \mu_n$ be the eigenvalues of the Laplacian edge dominating matrix $L_R(G)$

In Cauchy–Schwarz inequality, put $a_i = 1$ and $b_i = |\mu_i|$ we get,

$$\left(\sum_{i=1}^{n}|\mu_i|\right)^2 \le \left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}|\mu_i|^2\right)$$

Using Theorem 3.1.1, we have

$$\left(\sum_{i=1}^{n}|\mu_i|\right)^2 \le n\left(2|E| + \sum_{i=1}^{n}c_{v_i}^2\right)$$

$$\Rightarrow \left(\sum_{i=1}^{n}|\mu_i|\right) \le \sqrt{n\left(2|E| + \sum_{i=1}^{n}c_{v_i}^2\right)}$$

By Triangle inequality, $\left|\mu_i - \frac{2m}{n}\right| \le |\mu_i| + \left|\frac{2m}{n}\right| \quad \forall i = 1,2,3,\ldots n$

$$\Rightarrow \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right| \le \sum_{i=1}^{n}|\mu_i| + \sum_{i=1}^{n}\left|\frac{2m}{n}\right|$$

$$EL_R(G) \le \sqrt{n\left(2|E| + \sum_{i=1}^{n}c_{v_i}^2\right)} + 2m$$

$$\Rightarrow EL_R(G) \le \sqrt{n\left(2|E| + \sum_{i=1}^{n}c_{v_i}^2\right)} + 2|E|$$

3.2.3. Theorem (Upper bound).

Let G be a simple graph with n vertices, m size. Let $\mu_1, \mu_2, \ldots \mu_n$ be the eigenvalues of the reverse Laplacian matrix $L_R(G)$ then, $EL_R(G) \le \sqrt{2|E|\left(n - 2\sum_{i=1}^{n}c_{v_i}\right) + 4|E|^2 + n\sum_{i=1}^{n}c_{v_i}^2}$

Proof:

Let $\mu_1, \mu_2, \ldots \mu_n$ be the eigenvalues of the Laplacian edge dominating matrix $L_R(G)$.

In Cauchy–Schwarz inequality, put $a_i = 1$ and $b_i = \left|\mu_i - \frac{2m}{n}\right|$ we get,

$$\left(\sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|\right)^2 \le \left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|^2\right),$$

$$\begin{aligned}
(EL_R(G))^2 &\le n\left(\sum_{i=1}^{n}\mu_i^2 + \sum_{i=1}^{n}\frac{4m^2}{n^2} - \frac{4m}{n}\sum_{i=1}^{n}\mu_i\right) \\
&= n\left(2|E| + \sum_{i=1}^{n}c_{v_i}^2 + \frac{4m^2}{n} - \frac{4m}{n}\sum_{i=1}^{n}c_{v_i}\right) \\
&= 2n|E| + n\sum_{i=1}^{n}c_{v_i}^2 + 4m^2 - 4m\sum_{i=1}^{n}c_{v_i}
\end{aligned}$$

$$\Rightarrow EL_R(G) \le \sqrt{2n|E| + n\sum_{i=1}^{n}c_{v_i}^2 + 4|E|^2 - 4|E|\sum_{i=1}^{n}c_{v_i}}$$

$$\Rightarrow EL_R(G) \le \sqrt{2|E|\left(n - 2\sum_{i=1}^{n}c_{v_i}\right) + 4|E|^2 + n\sum_{i=1}^{n}c_{v_i}^2}$$

3.2.4. Theorem.

Let G be a graph with n vertices, m size. If the sum of the absolute eigenvalues of $L_R(G)$ is rational, then it must be an integer, satisfying the following relation:

$$\sum_{i=1}^{n} |\mu_i| \equiv \sum_{i=1}^{n} c_{v_i} \pmod 2$$

Proof:

Let $\mu_1, \mu_2, \dots \mu_n$ be the eigenvalues of the reverse Laplacian matrix $L_R(G)$such that $\mu_1, \mu_2, \dots \mu_t$ are positive and rest are non-positive. we have,

$$\begin{aligned}\sum_{i=1}^{n} |\mu_i| &= (\mu_1 + \mu_2 + \cdots + \mu_t) - (\mu_{(t+1)} + \mu_{(t+2)} + \cdots + \mu_n)\\ &= 2(\mu_1 + \mu_2 + \cdots + \mu_t) - (\mu_1 + \mu_2 + \cdots + \mu_n)\\ &= 2(\mu_1 + \mu_2 + \cdots + \mu_t) - \sum_{i=1}^{n} \mu_i\end{aligned}$$

By the result of, the partial sum $\mu_1 + \mu_2 + \cdots + \mu_t$ eigenvalues of a matrix whose characteristic polynomial has integer coefficients. If $\sum_{i=1}^{n} |\mu_i|$ is rational then $\mu_1 + \mu_2 + \cdots + \mu_t$ is rational, and hence it must be an integer.

$$\therefore \quad \sum_{i=1}^{n} |\mu_i| \equiv \sum_{i=1}^{n} c_{v_i} \pmod 2$$

4. Conclusions

In this paper, we had defined reverse Laplacian matrix, reverse Laplacian energy and studied the same for some standard graphs. We found upper for reverse Laplacian energy. Further, we proved if the sum of the absolute eigenvalues of $L_R(G)$ is rational then it must be an integer.

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Conflicts of Interest

The authors declare no conflict of interest.

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