Letters in Applied NanoBioScience
Platinum Open Access Journal (ISSN: 2284-6808)

Article

Volume 11, Issue 1, 2022, 3305 - 3321

https://doi.org/10.33263/LIANBS111.33053321

Topological Indices of Some Classes of Thorn Complete and Wheel Graphs

Divyashree B K [1], Jagadeesh R [2,*], Siddabasappa [3]

[1] Department of Mathematics, Government Science College, Bangalore University, Bengaluru 560001, India; divyashree1704@gmail.com (D.B.K.);
[2] Department of Mathematics, Government First Grade College, Ramanagar-562159, Karnataka, India; jagadeeshr1978@gmail.com (J.R.);
[3] Department of Mathematics, Government Science College, Bangalore University, Bengaluru 560001, India; siddabasappa1961@gmail.com (S.);
* Correspondence: jagadeeshr1978@gmail.com;

Received: 20.01.2021; Revised: 1.05.2021; Accepted: 5.05.2021; Published: 27.06.2021

Abstract: We have multiple real numbers that describe chemical descriptors in the field of Graph theory. These descriptors constitute the entire structure of a graph, which possesses an actual chemical structure. Among these, the main focus of topological indices is that they are associated with many non-identical physiochemical properties of chemical compounds. Also, the biological properties of chemical compounds can be established by the topological indices. In this analysis, we compute the Reciprocal Randic index(R^{-1}), Reduced Reciprocal Randic index(RR^{-1}), Atom-bond Connectivity index(ABC) and the geometric arithmetic index(GA) of thorn graphs are obtained theoretically.

Keywords: Reciprocal Randic(R^{-1}) index; Reduced Reciprocal Randic(RR^{-1}) index; Atom-bond Connectivity(ABC) index; Geometric-arithmetic(GA) index; cog-graph; thorn graph.

1. Introduction

We consider the connected graph $G = (V, E)$ with the cardinality of order p. Let $S = \{a_1, a_2, a_3, \dots, a_p\}$ be the p number of non-negative integers. The thorn graph G_s is acquired from the graph G by joining a_i number of pendent vertices to the vertex v_i of G for $i = 1,2, \dots p$. These fixed a_i pendent vertices to the vertices v_i of G are called thorns of G. Here we represent the set of a_i number of pendent vertices to the vertex v_i of G by V_k, $k = 1,2,3, \dots, p$. Thus$V(G_s) = V(G) \cup V_1(G) \cup V_2(G) \cup V_3(G) \cup \dots \cup V_p(G)$. In [1], Gutman et al., introduced the concept of thorn graphs. Along, with this, they established many applications related to chemistry [1, 2, 5-7].

The topological F_i the index was introduced to foresee the various temperature of the boiling point of hydrocarbons. There are two terms in the definition of F_i. in [2], Gutman et al., examined one of these terms, which is defined as follows.

Definition 1. Let $G = (V, E)$ be a connected graph and d_p denotes the degree of the vertex p, then RR^{-1} the index is defined as,

$$RR^{-1}(G) = \sum_{pq \in E(G)} \sqrt{(d_p - 1)(d_q - 1)}$$

In this paper, we discuss one of the currently established modified descriptions of Randic index, which is mentioned below.

Definition 2. Let G be the graph then the reciprocal randic index and is defined as

$$R^{-1}(G) = \sum_{pq\in E(G)} \sqrt{d_p d_q}$$

A topological index is a numerical value obtained mathematically in a distinct and very obvious manner from the given molecular graph. To design chemical compounds and its physio-chemical properties, it is mostly used in theoretical chemistry.

Definition 3. The topological index called the Atom-bond connectivity (ABC) index was introduced by Estrada et al., [2]. This index is defined as follows [1, 2]

$$ABC(G) = \sum_{pq\in E(G)} \sqrt{\frac{d_p + d_q - 2}{d_p d_q}}$$

Definition 4. For a molecular graph G, the geometric-arithmetic (GA) index is defined as [6]

$$GA(G) = \sum_{pq\in E(G)} \frac{2\sqrt{d_p d_q}}{d_p + d_q}$$

2. Materials and Methods

We collect all the information's and materials for our work digitally. The main focus of this paper is to obtain topological indices of standard thorn graphs and some of its other classes. Firstly, we identify the degree of each graph's vertices; later, we partition the edges of the graph, which follow the same property to procure our results.

3. Results and Discussion

In this portion of the paper, we found results related to the Reciprocal Randic index(R^{-1}), Reduced Reciprocal Randic index(RR^{-1}), Atom-bond Connectivity index(ABC) and the geometric arithmetic index(GA) of thorn graphs and its special cases.

3.1. Degree-based indices for standard graphs and some classes of its thorn graphs.

3.1.1. The complete graph K_a.

A complete graph K_a is a graph in which every pair of vertices is adjacent. The graph K_a has a number of vertices with $\frac{a(a-1)}{2}$ number of edges.

3.1.2. Thorn of the complete graph K_a.

The b-thorn complete graph $K_{a,b}$ has a complete graph K_a as the parent and $(b-a)$ thorns that is u_i pendent vertices, $i = 1,2,\dots,b$ at each vertex v_i for $i = 1,2,\dots,a$ of K_a where $(a, b > 2)$. The b-thorn complete graph $K_{a,b}$ is regarded as the thorn graph $(K_a)_S$ where $S = (u_1, u_2, \dots, u_b)$. Then the number of vertices of$K_{a,b}$, is $p = a + \sum_{i=1}^{b} u_i$ and the number of edges are $q = \frac{n(n-1)}{2} + \sum_{i=1}^{b} u_i$. The b-thorn complete graph $K_{a,b}$ is shown in Figure 1.

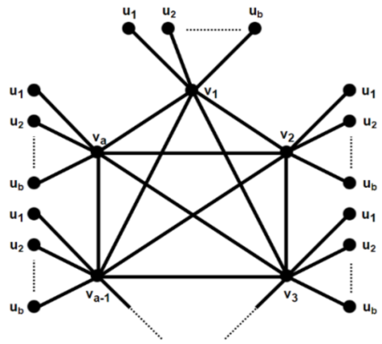

Figure 1. Thorn complete graph $K_{a,b}$.

Theorem 3.1. The reciprocal randic index of a thorn complete graph having $(a + ba)$ number of vertices is

$$R^{-1}(K_{a,b}) = a_i b_j \sqrt{(b_j + a - 1)} + a_i(b_j + a - 1).$$

Proof: Here, we independently examine the pair of vertices of K_a between the pair of vertices u_i for $i = 1,2,3, \dots, b$ of $K_{a,b}$ along with the pair of vertices that belongs to K_a and otherwise degree 1 vertex. Let $v_1, v_2, v_3, \dots, v_a$ be the vertices of the complete graph K_a where $i = 2,3,4, \dots, a$ and let $u_1, u_2, u_3, \dots, u_b$ be a degree one vertices of K_a where $j = 1,2,3, \dots, b$. (As in Figure 1). Let $d(v_i) = (b_j + a - 1)$ and $d(u_j) = 1$. Then we have,

$$R^{-1}(K_{a,b}) = \sum_{i,j=1}^{a_i,b_j} \sqrt{d(v_i)d(u_j)} + \sum_{i=1}^{a_i} \sqrt{d(v_i)d(v_i)}$$
$$= \sum_{i,j=1}^{a_i,b_j} \sqrt{(b_j + a - 1)1} + \sum_{i=1}^{a_i} \sqrt{(b_j + a - 1)(b_j + a - 1)}$$
$$= a_i b_j \sqrt{(b_j + a - 1)} + a_i \sqrt{(b_j + a - 1)(b_j + a - 1)}$$
$$= a_i b_j \sqrt{(b_j + a - 1)} + a_i(b_j + a - 1).$$

Theorem 3.2: The reduced reciprocal randic index of a thorn complete graph having $(a + ab)$ number of vertices is

$$RR^{-1}(K_{a,b}) = a_i(b_j + a - 2).$$

Proof: Here we independently examine the pair of vertices of K_a between the pair of vertices u_i for $i = 1,2,3, \dots, b$ of $K_{a,b}$ along with the pair of vertices that belongs to K_a and otherwise degree 1 vertex. Let $v_1, v_2, v_3, \dots, v_a$ be the vertices of the complete graph K_a where $i = 2,3,4, \dots, a$ and let $u_1, u_2, u_3, \dots, u_b$ be a degree one vertices of K_a where $j = 1,2,3, \dots, b$. (As in Figure 1). Let $d(v_i) = (b_j + a - 1)$ and $d(u_j) = 1$. Then we have,

$$RR^{-1}(K_{a,b}) = \sum_{i,j=1}^{a_i,b_j} \sqrt{(d(v_i) - 1)(d(u_j) - 1)} + \sum_{i=1}^{a_i} \sqrt{(d(v_i) - 1)(d(v_i) - 1)}$$
$$= \sum_{i,j=1}^{a_i,b_j} \sqrt{(b_j + a - 1)(1 - 1)} +$$
$$\sum_{i=1}^{a_i} \sqrt{(b_j + a - 1 - 1)(b_j + a - 1 - 1)}$$
$$= 0 + a_i \sqrt{(b_j + a - 2)(b_j + a - 2)}$$
$$= a_i(b_j + a - 2).$$

Theorem 3.3: The Atom-bond connectivity index of a thorn complete graph having $(a + ab)$ number of vertices is

$$ABC(K_{a,b}) = a_i b_j \sqrt{\frac{(b_j + a - 2)}{(b_j + a - 1)}} + a_i \sqrt{\frac{2b_j + 2(a-2)}{(b_j + a - 1)^2}}$$

Proof: Here, we independently examine the pair of vertices of K_a between the pair of vertices u_i for $i = 1,2,3, \dots, b$ of $K_{a,b}$ along with the pair of vertices that belongs to K_a and otherwise degree 1 vertex. Let $v_1, v_2, v_3, \dots, v_a$ be the vertices of the complete graph K_a where $i = 2,3,4, \dots, a$ and let $u_1, u_2, u_3, \dots, u_b$ be a degree one vertices of K_a where$j = 1,2,3, \dots, b$. (As in Figure 1). Let $d(v_i) = (b_j + a - 1)$ and $d(u_j) = 1$. Then we have,

$$ABC(K_{a,b}) = \sum_{i,j=1}^{a_i,b_j} \sqrt{\frac{d(u_j)+d(v_i)-2}{d(u_j)\cdot d(v_i)}} + \sum_{i=1}^{a_i} \sqrt{\frac{d(v_i)+d(v_i)-2}{d(v_i)\cdot d(v_i)}}$$

$$= \sum_{i,j=1}^{a_i,b_j} \sqrt{\frac{(1)+(b_j+a-1)-2}{(1)(b_j+a-1)}} + \sum_{i=1}^{a_i} \sqrt{\frac{(b_j+a-1)+(b_j+a-1)-2}{(b_j+a-1)(b_j+a-1)}}$$

$$= a_i b_j \sqrt{\frac{(b_j+a-2)}{(b_j+a-1)}} + a_i \sqrt{\frac{2b_j+2(a-2)}{(b_j+a-1)^2}}.$$

Theorem 3.4: The geometric-arithmetic index of a thorn complete graph having $(a + ab)$ number of vertices is

$$GA(K_{a,b}) = a_i b_j \frac{2\sqrt{(b_j+a-1)}}{(b_j+a)} + a_i.$$

Proof: Here, we independently examine the pair of vertices of K_a between the pair of vertices u_i for $i = 1,2,3, \dots, b$ of $K_{a,b}$ along with the pair of vertices that belongs to K_a and otherwise degree 1 vertex. Let $v_1, v_2, v_3, \dots, v_a$ be the vertices of the complete graph K_a where $i = 2,3,4, \dots, a$ and let $u_1, u_2, u_3, \dots, u_b$ be a degree one vertices of K_a where $j = 1,2,3, \dots, b$. (As in Figure.1). Let $d(v_i) = (b_j + a - 1)$ and $d(u_j) = 1$. Then we have,

$$GA(K_{a,b}) = \sum_{i,j=1}^{a_i,b_j} \frac{2\sqrt{d(v_i)\cdot d(u_j)}}{d(v_i)+d(u_j)} + \sum_{i=1}^{a_i} \frac{2\sqrt{d(v_i)d(v_i)}}{d(v_i)+d(v_i)}$$

$$= \sum_{i,j=1}^{a_i,b_j} \frac{2\sqrt{(b_j+a-1)}}{(b_j+a)} + \sum_{i=1}^{a_i} \frac{2\sqrt{(b_j+a-1)^2}}{(b_j+a-1+b_j+a-1)}$$

$$= a_i b_j \frac{2\sqrt{(b_j+a-1)}}{(b_j+a)} + a_i.$$

3.1.3. A cog-complete graph $K^c{}_a$.

$K^c{}_a$ is the graph formed from a complete graph K_a $(a \geq 2)$of the vertex set $\{v_1, v_2, v_3, \dots, v_a\}$ with the addition of b number of vertices such as $\{u_1, u_2, u_3, \dots, u_b\}$ and $2b$ number of edges given by $\{u_j v_i, u_j v_{i+1}: i = 1,2,3, \dots, a \text{ and } j = 1,2,3, \dots, b\}(v_{a+1} = v_1)$, as shown in Figure 2.

We know that $p(K^c{}_a) = 2(a + b), q(K^c{}_a) = \frac{a(a+3)}{2}$.

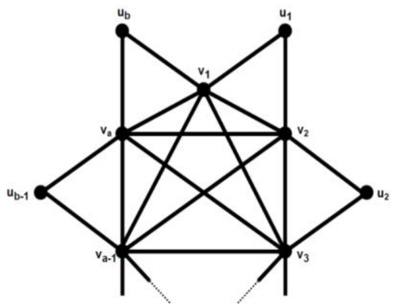

Figure 2. Cog-Complete graph $K^c{}_a$.

3.1.4. A thorn cog-complete graph $K^{c^*}{}_a$

$K^{c^*}{}_a$ is the cog-complete graph $K^c{}_a$ $(a \geq 2)$ is obtained from Definition 3. , with $2b$ number of additional vertices of degree 1 such as $\{w_1, w_2, w_3, \ldots\ldots, w_{2b}\}$ for $k = 1,2,3, \ldots\ldots, 2b$ and edges given by $\{u_j w_{2k-1}, u_j w_{2k} : i, j = 1,2,3, \ldots\ldots, b\}$, as shown in Figure 3.

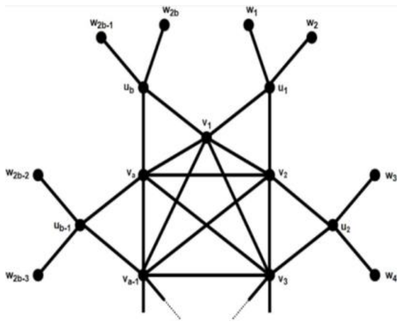

Figure 3. Thorn Cog-Complete graph $K^{c^*}{}_{b,a}$.

Theorem 3.5: The reciprocal randic index of a thorn cog-complete graph having $(a + 3b)$ number of vertices is

$$R^{-1}\left(K^{c^*}{}_{b,a}\right) = 2a_i b_j \sqrt{(a+1)} + a_i(a+1) + 4b_j b_k.$$

Proof: Here, we independently examine the pair of vertices of K_a between the pair of vertices u_i for $i = 1,2,3, \ldots, b$ of $K^c{}_a$ along with the pair of vertices that belongs to K_a and between the pair of vertices u_i, for $i = 1,2,3, \ldots, b$ of $K^c{}_a$ along with the pair of vertices degree 1 vertices of $K^{c^*}{}_{b,a}$. Let $v_1, v_2, v_3, \ldots., v_a$ be the vertices of the complete graph K_a where $i = 2,3,4, \ldots, a$ and let $u_1, u_2, u_3, \ldots, u_b$ be the addition of b number of vertices of $K^c{}_a$ where $j = 1,2,3, \ldots., b$. Also, let $w_1, w_2, \ldots\ldots, w_{2b}, k = 1,2,3, \ldots., 2b$ of $K^{c^*}{}_{b,a}$ be an additional $2b$ number of degree 1 vertices of $K^{c^*}{}_{b,a}$. (As in Figure. 3). Let $d(v_i) = (a+1)$, $d(u_j) = 4$ and $d(w_k) = 1$. Then we have,

$$R^{-1}\left(K^{c^*}{}_{b,a}\right) = \sum_{i,j=1}^{a_i,b_j} \sqrt{d(v_i)d(u_j)} + \sum_{i=1}^{a_i} \sqrt{d(v_i)d(v_i)} + \sum_{j,k=1}^{b_j,2b_k} \sqrt{d(u_j)d(w_k)}$$

$$= \sum_{i,j=1}^{a_i,b_j} \sqrt{(a+1)4} + \sum_{i=1}^{a_i} \sqrt{(a+1)(a+1)} + \sum_{j,k=1}^{b_j,2b_k} \sqrt{1(4)}$$

$$= a_i b_j \sqrt{4(a+1)} + a_i \sqrt{(a+1)(a+1)} + b_j(2b_k)2$$

$$= 2a_i b_j \sqrt{(a+1)} + a_i(a+1) + 4b_j b_k.$$

Theorem 3.6: The reduced reciprocal randic index of a thorn cog-complete graph having $(a + 3b)$ number of vertices is

$$RR^{-1}\left(K^{c^*}{}_{b,a}\right) = a_i b_j \sqrt{3a} + a_i(a).$$

Proof: Here, we independently examine the pair of vertices of K_a between the pair of vertices u_i for $i = 1,2,3, \ldots, b$ of $K^c{}_a$ along with the pair of vertices that belongs to K_a and between the pair of vertices u_i for $i = 1,2,3, \ldots, b$ of $K^c{}_a$ along with the pair of vertices degree 1 vertices of $K^{c^*}{}_{b,a}$. Let $v_1, v_2, v_3, \ldots., v_a$ be the vertices of the complete graph K_a where $i = 2,3,4, \ldots, a$ and let $u_1, u_2, u_3, \ldots, u_b$ be the addition of b number of vertices of $K^c{}_a$ where $j = 1,2,3, \ldots., b$. Also, let $w_1, w_2, \ldots\ldots, w_{2b}, k = 1,2,3, \ldots., 2b$ of $K^{c^*}{}_{b,a}$ be an additional $2b$ number of degree 1 vertices of $K^{c^*}{}_{b,a}$. (As in Figure. 3). Let $d(v_i) = (a+1)$, $d(u_j) = 4$ and $d(w_k) = 1$. Then we have,

$$RR^{-1}\left(K^{c^*}{}_{b,a}\right) = \sum_{i,j=1}^{a_i,b_j} \sqrt{(d(v_i)-1)(d(u_j)-1)} + \sum_{i=1}^{a_i} \sqrt{(d(v_i)-1)^2} + \sum_{j,k=1}^{b_j,2b_k} \sqrt{(d(u_j)-1)(d(w_k)-1)}$$

$$= \sum_{i,j=1}^{a_i,b_j} \sqrt{(a+1-1)(4-1)} + \sum_{i=1}^{a_i} \sqrt{(a+1-1)(a+1-1)} + \sum_{j,k=1}^{b_j,2b_k} \sqrt{(1-1)(4-1)}$$

$$= a_i b_j \sqrt{3a} + a_i \sqrt{a(a)}$$

$$= a_i b_j \sqrt{3a} + a_i (a).$$

Theorem 3.7: The Atom-bond connectivity index of a thorn cog-complete graph having $(a+3b)$ number of vertices is

$$ABC\left(K^{c^*}{}_{b,a}\right) = a_i b_j \sqrt{\frac{a+3}{4(a+1)}} + a_i \frac{\sqrt{2a}}{(a+1)} + b_j\, b_k \sqrt{3}\,.$$

Proof: Here, we independently examine the pair of vertices of K_a between the pair of vertices u_i for $i = 1,2,3,\dots,b$ of $K^c{}_a$ along with the pair of vertices that belongs to K_a and between the pair of vertices u_i for $i = 1,2,3,\dots,b$ of $K^c{}_a$ along with the pair of vertices degree 1 vertices of $K^{c^*}{}_{b,a}$. Let $v_1, v_2, v_3, \dots, v_a$ be the vertices of the complete graph K_a where $i = 2,3,4,\dots,a$ and let $u_1, u_2, u_3, \dots, u_b$ be the addition of b number of vertices of $K^c{}_a$ where $j = 1,2,3,\dots,b$. Also, let $w_1, w_2, \dots, w_{2b}, k = 1,2,3,\dots,2b$ of $K^{c^*}{}_{b,a}$, be an additional $2b$ number of degree 1 vertices of $K^{c^*}{}_{b,a}$. (As in Figure. 3). Let $d(v_i) = (a+1)$, $d(u_j) = 4$ and $d(w_k) = 1$. Then we have,

$$ABC\left(K^{c^*}{}_{b,a}\right) = \sum_{i,j=1}^{a_i,b_j} \sqrt{\frac{d(v_i)+d(u_j)-2}{d(v_i)d(u_j)}} + \sum_{i=1}^{a_i} \sqrt{\frac{d(v_i)+d(v_i)-2}{d(v_i)d(v_i)}} + \sum_{j,k=1}^{b_j,2b_k} \sqrt{\frac{d(u_j)+d(w_k)-2}{d(u_j)d(w_k)}}$$

$$= \sum_{i,j=1}^{a_i,b_j} \sqrt{\frac{a+1+4-2}{4(a+1)}} + \sum_{i=1}^{a_i} \sqrt{\frac{(a+1)+(a+1)-2}{(a+1)^2}} + \sum_{j,k=1}^{b_j,2b_k} \sqrt{\frac{4+1-2}{(4)1}}$$

$$= a_i b_j \sqrt{\frac{a+3}{4(a+1)}} + a_i \sqrt{\frac{2a}{(a+1)^2}} + b_j (2b_k) \sqrt{\frac{3}{(4)}}$$

$$= a_i b_j \sqrt{\frac{a+3}{4(a+1)}} + a_i \frac{\sqrt{2a}}{(a+1)} + b_j\, b_k \sqrt{3}\,.$$

Theorem 3.8: The geometric-arithmetic index of a thorn cog-complete graph having $(a+3b)$ number of vertices is

$$GA\left(K^{c^*}{}_{b,a}\right) = a_i b_j \frac{4\sqrt{(a+1)}}{(a+5)} + a_i + \frac{8 b_j\, b_k}{5}.$$

Proof: Here we independently examine the pair of vertices of K_a between the pair of vertices u_i for $i = 1,2,3,\dots,b$ of $K^c{}_a$ along with the pair of vertices which belongs to K_a and between the pair of vertices u_i for $i = 1,2,3,\dots,b$ of $K^c{}_a$ along with the pair of vertices degree 1 vertices of $K^{c^*}{}_{b,a}$. Let $v_1, v_2, v_3, \dots, v_a$ be the vertices of the complete graph K_a where $i =$

$2,3,4,\dots,a$ and let $u_1,u_2,u_3,\dots,u_b$ be addition of b number of vertices of $K^c{}_a$ where $j = 1,2,3,\dots,b$. Also, let $w_1,w_2,\dots\dots,w_{2b}, k = 1,2,3,\dots,2b$ of $K^{c^*}{}_{b,a}$ be an additional $2b$ number of degree 1 vertices of $K^{c^*}{}_{b,a}$. (As in Figure. 3). Let $d(v_i) = (a+1)$, $d(u_j) = 4$ and $d(w_k) = 1$. Then we have,

$$GA(K^{c^*}{}_{b,a}) = \sum_{i,j=1}^{a_i,b_j} \frac{2\sqrt{d(v_i)d(u_j)}}{d(v_i)+d(u_j)} + \sum_{i=1}^{a_i} \frac{2\sqrt{d(v_i)d(v_i)}}{d(v_i)+d(v_i)} + \sum_{j,k=1}^{b_j,2b_k} \frac{2\sqrt{d(u_j)d(w_k)}}{d(u_j)+d(w_k)}$$

$$= \sum_{i,j=1}^{a_i,b_j} \frac{2\sqrt{4(a+1)}}{(a+1+4)} + \sum_{i=1}^{a_i} \frac{2\sqrt{(a+1)(a+1)}}{(a+1)+(a+1)} + \sum_{j,k=1}^{b_j,2b_k} \frac{2\sqrt{4(1)}}{(5)}$$

$$= a_i b_j \frac{4\sqrt{(a+1)}}{a+5} + a_i \frac{2\sqrt{(a+1)^2}}{2a+2} + b_j(2b_k)\ \frac{4}{5}$$

$$= a_i b_j \frac{4\sqrt{(a+1)}}{a+5} + a_i + \frac{8b_j\, b_k}{5}.$$

3.1.5. A cog-star graph $S^c{}_a$.

$S^c{}_a$ is the graph formed from a star graph S_a $(a \geq 4)$, of the vertex set $\{v_1, v_2, v_3, \dots, v_{a-1}, v_a\}$ with the addition of $(b-1)$ number of vertices such as $\{u_1, u_2, u_3, \dots\dots, u_{b-1}\}$ and $2b$ number of edges given by $\{u_j v_{i+1}, u_j v_{i+2} : i = 1,2,3,\dots,a \text{ and } j = 1,2,3,\dots\dots,(b-1)\}$ $(v_{a+1} = v_2)$, as shown in Figure 4.
We know that $p(S^c{}_a) = (a+b-1), q(S^c{}_a) = a+2b-3$.

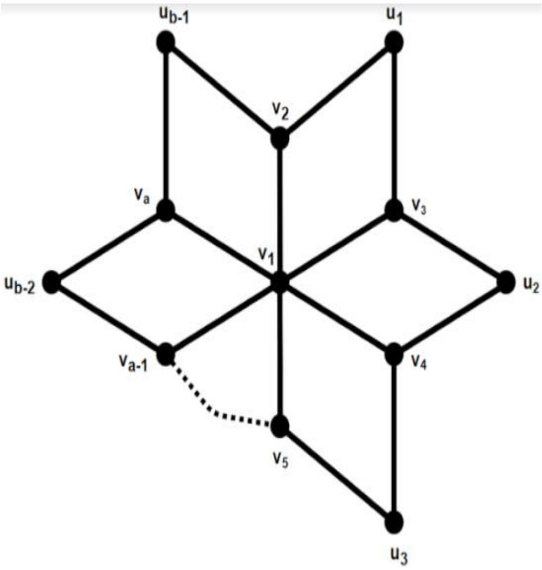

Figure 4. A Cog-Star graph S_a^c.

3.1.6. The thorn Cog-Star graph $S^{c^*}{}_a$.

$S^{c^*}{}_a$ is the graph formed from a star graph S_a^c ($a \geq 4$), of the vertex set $\{v_1, v_2, v_3, \dots, v_{a-1}, v_a, u_1, u_2, u_3, \dots\dots, u_{b-1}\}$ for $i = 1,2,3,\dots,a$ and $j = 1,2,3,\dots\dots,(b-1)$with the addition of $2(b-1)$ number of vertices such as $\{w_1, w_2, w_3, \dots\dots, w_{2b-3}, w_{2b-2}\}$ and edges given by $\{u_j w_{2j-1}, u_j w_{2j} : j = 1,2,3,\dots\dots,(b-1)\}$, as shown in Figure 5.

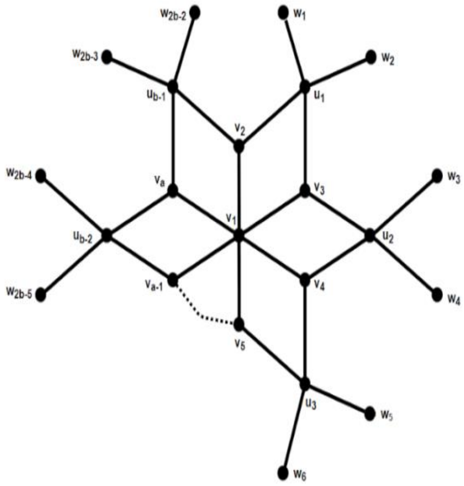

Figure 5. Thorn Cog-Star graph $S^{c^*}{}_a$.

Theorem 3.9: The reciprocal randic index of a thorn cog-star graph having $(a+3(b-1))$ number of vertices is

$$R^{-1}(S^{c^*}{}_a) = 2a_i(b_j-1)\sqrt{3} + a_i\sqrt{3(a-1)} + 4(b_j-1)(b_k-1).$$

Proof: Here, we independently examine the pair of vertices of S_a between the pair of vertices u_j for $j = 1,2,3,\dots,(b-1)$ of $S^c{}_a$ along with the pair of vertices that belongs to S_a and between the pair of vertices u_j for $j = 1,2,3,\dots,(b-1)$ of $S^c{}_a$ along with the pair of vertices degree 1 vertices of $S^{c^*}{}_a$. Let $v_1, v_2, v_3, \dots, v_a$ be the vertices of a star graph S_a where $i\ ranges\ from$ $1,2,3,4,\dots,a$ and let $u_1, u_2, u_3, \dots, u_{b-1}$ be the addition of $(b-1)$ number of vertices of $S^c{}_a$ where $j = 1,2,3,\dots,(b-1)$. Also, let $w_1, w_2, \dots\dots, w_{2b-2}$, $k = 1,2,3,\dots,(2b-2)$ of $S^{c^*}{}_a$ be an additional $(2b-2)$ number of degree 1 vertices of $S^{c^*}{}_a$. (As in Figure. 5). Let $d(v_1) = (a-1)$, $d(u_j) = 4, d(v_i) = 3, i \in \{2,3,4,\dots,a\}$ and $d(w_k) = 1$. Then we have,

$$R^{-1}(S^{c^*}{}_a) = \sum_{i=2,j=1}^{a_i,(b_j-1)} \sqrt{d(v_i)d(u_j)} + \sum_{i=2}^{a_i} \sqrt{d(v_1)d(v_i)}$$

$$+\sum_{j,k=1}^{b_j-1,2b_k-2} \sqrt{d(u_j)d(w_k)}$$

$$= \sum_{i=2,j=1}^{a_i,(b_j-1)} \sqrt{(3)4} + \sum_{i=2}^{a_i} \sqrt{(a-1)3} + \sum_{j,k=1}^{b_j-1,2b_k-2} \sqrt{1(4)}$$

$$= a_i(b_j-1)\sqrt{4(3)} + a_i\sqrt{(a-1)(3)} + (b_j-1)(2b_k-2)2$$

$$= 2a_i(b_j-1)\sqrt{3} + a_i\sqrt{3(a-1)} + 4(b_j-1)(b_k-1).$$

Theorem 3.10: The reduced reciprocal randic index of a thorn cog-star graph having $(a+3(b-1))$ number of vertices is

$$RR^{-1}(S^{c^*}{}_a) = a_i(b_j-1)\sqrt{6} + a_i\sqrt{2(a-2)}.$$

Proof: Here, we independently examine the pair of vertices of S_a between the pair of vertices u_j for $j = 1,2,3,\dots,(b-1)$ of $S^c{}_a$ along with the pair of vertices that belongs to S_a and between the pair of vertices u_j for $j = 1,2,3,\dots,(b-1)$ of $S^c{}_a$ along with the pair of vertices degree 1 vertices of $S^{c^*}{}_a$. Let $v_1, v_2, v_3, \dots, v_a$ be the vertices of a star graph S_a where $i\ ranges\ from$ 1,2,3,4, …, a and let $u_1, u_2, u_3, \dots, u_{b-1}$ be the addition of $(b-1)$ number of vertices of $S^c{}_a$ where $j = 1,2,3, \dots, (b-1)$. Also, let $w_1, w_2, \dots, w_{2b-2}$, $k = 1,2,3, \dots, (2b-2)\ of\ S^{c^*}{}_a$ be an additional $(2b-2)$ number of degree 1 vertices of $S^{c^*}{}_a$. (As in Figure. 5). Let $d(v_1) = (a-1)$, $d(u_j) = 4, d(v_i) = 3, i \in \{2,3,4, \dots, a\}$ and $d(w_k) = 1$. Then we have,

$$RR^{-1}(S^{c^*}{}_a) = \sum_{i=2,j=1}^{a_i,(b_j-1)} \sqrt{(d(v_i)-1)(d(u_j)-1)} + \sum_{i=2}^{a_i} \sqrt{(d(v_1)-1)(d(v_i)-1)} +$$

$$\sum_{j,k=1}^{b_j-1,2b_k-2} \sqrt{(d(u_j)-1)(d(w_k)-1)}$$

$$= \sum_{i=2,j=1}^{a_i,(b_j-1)} \sqrt{(3-1)(4-1)} + \sum_{i=2}^{a_i} \sqrt{(a-1-1)(3-1)}$$

$$+\sum_{j,k=1}^{b_j-1,2b_k-2} \sqrt{(1-1)(4-1)}$$

$$= a_i(b_j-1)\sqrt{6} + a_i\sqrt{2(a-2)}.$$

Theorem 3.11: The Atom-bond connectivity index of a thorn cog-star graph having $(a+3(b-1))$ number of vertices is

$$ABC(S^{c^*}{}_a) = a_i(b_j-1)\sqrt{\frac{5}{12}} + a_i\sqrt{\frac{a}{3(a-1)}} + (b_j-1)(b_k-1)\sqrt{3}\,.$$

Proof: Here, we independently examine the pair of vertices of S_a between the pair of vertices u_j for $j = 1,2,3, \dots, (b-1)$ of $S^c{}_a$ along with the pair of vertices that belongs to S_a and between the pair of vertices u_j for $j = 1,2,3, \dots, (b-1)$ of $S^c{}_a$ along with the pair of vertices degree 1 vertices of $S^{c^*}{}_a$. Let $v_1, v_2, v_3, \dots, v_a$ be the vertices of a star graph S_a where $i\ ranges\ from$ 1,2,3,4, …, a and let $u_1, u_2, u_3, \dots, u_{b-1}$ be the addition of $(b-1)$ number of vertices of $S^c{}_a$ where $j = 1,2,3, \dots, (b-1)$. Also, let $w_1, w_2, \dots, w_{2b-2}$, $k = 1,2,3, \dots, (2b-2)\ of\ S^{c^*}{}_a$ be an additional $(2b-2)$ number of degree 1 vertices of $S^{c^*}{}_a$. (As in Figure. 5). Let $d(v_1) = (a-1)$, $d(u_j) = 4, d(v_i) = 3, i \in \{2,3,4, \dots, a\}$ and $d(w_k) = 1$. Then we have,

$$ABC(S^{c^*}{}_a) = \sum_{i=2,j=1}^{a_i,(b_j-1)} \sqrt{\frac{d(v_i)+d(u_j)-2}{d(v_i)d(u_j)}} +$$

$$\sum_{i=2}^{a_i} \sqrt{\frac{d(v_1)+d(v_i)-2}{d(v_1)d(v_i)}} + \sum_{j,k=1}^{b_j-1,2b_k-2} \sqrt{\frac{d(u_j)+d(w_k)-2}{d(u_j)d(w_k)}}$$

$$= \sum_{i=2,j=1}^{a_i,(b_j-1)} \sqrt{\frac{3+4-2}{3(4)}} + \sum_{i=2}^{a_i} \sqrt{\frac{(a-1)+(3)-2}{3(a-1)}} + \sum_{j,k=1}^{b_j-1,2b_k-2} \sqrt{\frac{4+1-2}{(4)(1)}}$$

$$= a_i(b_j-1)\sqrt{\frac{5}{4(3)}} + a_i\sqrt{\frac{a}{3(a-1)}} + (b_j-1)(2b_k-2)\sqrt{\frac{3}{(4)}}$$

$$= a_i(b_j-1)\sqrt{\frac{5}{12}} + a_i\sqrt{\frac{a}{3(a-1)}} + (b_j-1)(b_k-1)\sqrt{3}\,.$$

Theorem 3.12: The geometric-arithmetic index of a thorn cog-star graph having $(a + 3(b-1))$ number of vertices is

$$GA(S^{c^*}{}_a) = a_i(b_j-1)\frac{4\sqrt{3}}{7} + a_i\frac{2\sqrt{3(a-1)}}{(a+2)} + \frac{8(b_j-1)(b_k-1)}{5}.$$

Proof: Here, we independently examine the pair of vertices of S_a between the pair of vertices u_j for $j = 1,2,3,\dots,(b-1)$ of $S^c{}_a$ along with the pair of vertices that belongs to S_a and between the pair of vertices u_j for $j = 1,2,3,\dots,(b-1)$ of $S^c{}_a$ along with the pair of vertices degree 1 vertices of $S^{c^*}{}_a$. Let $v_1, v_2, v_3, \dots, v_a$ be the vertices of a star graph S_a where $i\ ranges\ from$ $1,2,3,4,\dots,a$ and let $u_1, u_2, u_3, \dots, u_{b-1}$ be the addition of $(b-1)$ number of vertices of $S^c{}_a$ where $j = 1,2,3,\dots,(b-1)$. Also, let $w_1, w_2, \dots, w_{2b-2}$, $k = 1,2,3,\dots,(2b-2)$ of $S^{c^*}{}_a$ be an additional $(2b-2)$ number of degree 1 vertices of $S^{c^*}{}_a$. (As in Figure. 5). Let $d(v_1) = (a-1)$, $d(u_j) = 4, d(v_i) = 3, i = 2,3,4,\dots,a$ and $d(w_k) = 1$. Then we have,

$$GA(S^{c^*}{}_a) = \sum_{i=2,j=1}^{a_i,(b_j-1)} \frac{2\sqrt{d(v_i)d(u_j)}}{(d(v_i)+d(u_j))} + \sum_{i=2}^{a_i} \frac{2\sqrt{d(v_1)d(v_i)}}{(d(v_1)+d(v_i))}$$

$$+ \sum_{j,k=1}^{b_j-1,2b_k-2} \frac{2\sqrt{d(u_j)d(w_k)}}{(d(u_j)+d(w_k))}$$

$$= \sum_{i=2,j=1}^{a_i,(b_j-1)} \frac{2\sqrt{3(4)}}{(3+4)} + \sum_{i=2}^{a_i} \frac{2\sqrt{3(a-1)}}{((a-1)+3)} + \sum_{j,k=1}^{b_j-1,2b_k-2} \frac{2\sqrt{1(4)}}{(4+1)}$$

$$= a_i(b_j-1)\frac{4\sqrt{3}}{7} + a_i\frac{2\sqrt{3(a-1)}}{(a+2)} + (b_j-1)(2b_k-2)\frac{4}{5}$$

$$= a_i(b_j-1)\frac{4\sqrt{(3)}}{(7)} + a_i\frac{2\sqrt{3(a-1)}}{(a+2)} + \frac{8(b_j-1)(b_k-1)}{5}.$$

3.1.7. Thorn of wheel graph W_a.

The b-thorn wheel graph $W_{a,b}$ has a wheel graph W_a as the parent, and $(b-3)$ thorns that are u_i pendent vertices $i = 1,2,\dots,b$, at each vertex $v_i\ for\ i = 1,2,\dots,a$ of W_a where $a, b > 3$. The b-thorn wheel graph $W_{a,b}$ is regarded as the thorn graph $(W_a)_S$ where $S = (u_1, u_2, \dots, u_b)$. Then the number of vertices of $W_{a,b}$ is $p = a + \sum_{i=1}^{b} u_i$ and the number of edges are $q = 2(a-1) + \sum_{i=1}^{b} u_i$. The b-thorn wheel graph $W_{a,b}$ is shown in Figure 6.

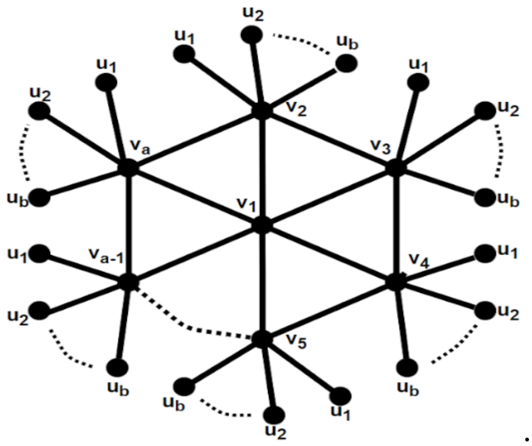

Figure 6. Thorn Wheel graph $W_{a,b}$.

Theorem 3.13: The reciprocal randic index of a thorn wheel graph having $(a + (a-1)b)$ number of vertices is

$$R^{-1}(W_{a,b}) = a_i b_j \sqrt{(b+3)} + a_i(b+3) + a_i\sqrt{(a-1)(b+3)}.$$

Proof: Here, we independently examine the pair of vertices of W_a between the pair of vertices u_i for $i = 1,2,3,\dots,b$ of $W_{a,b}$ along with the pair of vertices that belongs to W_a and otherwise degree 1 vertex. Let $v_1, v_2, v_3, \dots, v_a$ be the vertices of wheel graph W_a where $i\ ranges\ from\ 1,2,3,4,\dots,a$ and let $u_1, u_2, u_3, \dots, u_b$ be a degree one vertices of $W_{a,b}$ where$j = 1,2,3,\dots,b$. (As in Figure. 6). Let $d(v_i) = (b+3)\ for\ i = 2,3,4,\dots.a, d(v_1) = (a-1)$ and $d(u_j) = 1$. Then we have,

$$R^{-1}(W_{a,b}) = \sum_{i=2,j=1}^{a_i,b_j} \sqrt{d(v_i)d(u_j)} + \sum_{i=2}^{a_i} \sqrt{d(v_i)d(v_i)}$$

$$+\sum_{i=2}^{a_i} \sqrt{d(v_1)d(v_i)}$$

$$= \sum_{i=2,j=1}^{a_i,b_j} \sqrt{(b+3)1} + \sum_{i=2}^{a_i} \sqrt{(b+3)(b+3)}$$

$$+\sum_{i=2}^{a_i} \sqrt{(a-1)(b+3)}$$

$$= a_i b_j \sqrt{(b+3)} + a_i\sqrt{(b+3)^2} + a_i\sqrt{(a-1)(b+3)}$$

$$= a_i b_j \sqrt{(b+3)} + a_i(b+3) + a_i\sqrt{(a-1)(b+3)}$$

Theorem 3.14: The reduced reciprocal randic index of a thorn wheel graph having $(a + (a-1)b)$ number of vertices is

$$RR^{-1}(W_{a,b}) = a_i(b+2) + a_i\sqrt{(a-2)(b+2)}.$$

Proof: Here, we independently examine the pair of vertices of W_a between the pair of vertices u_i for $i = 1,2,3,\dots,b$ of $W_{a,b}$ along with the pair of vertices that belongs to W_a and otherwise degree 1 vertex. Let $v_1, v_2, v_3, \ldots., v_a$ be the vertices of wheel graph W_a where $i\ ranges\ from$ 1,2,3,4, …, a and let $u_1, u_2, u_3, \ldots, u_b$ be a degree one vertices of $W_{a,b}$ where $j = 1,2,3, \ldots., b$. (As in Figure. 6). Let $d(v_i) = (b+3)\ for\ i = 2,3,4, \ldots. a, d(v_1) = (a-1)$ and $d(u_j) = 1$. Then we have,

$$RR^{-1}(W_{a,b}) = \sum_{i=2,j=1}^{a_i,b_j} \sqrt{(d(v_i)-1)(d(u_j)-1)} + \sum_{i=2}^{a_i} \sqrt{(d(v_i)-1)^2}$$

$$+\sum_{i=2}^{a_i} \sqrt{(d(v_1)-1)(d(v_i)-1)}$$

$$= \sum_{i=2,j=1}^{a_i,b_j} \sqrt{(b+3-1)(1-1)} + \sum_{i=2}^{a_i} \sqrt{(b+3-1)^2}$$

$$+\sum_{i=2}^{a_i} \sqrt{(a-2)(b+3-1)}$$

$$= a_i(b+2) + a_i\sqrt{(a-2)(b+2)}.$$

Theorem 3.15: The Atom-bond connectivity index of a thorn wheel graph having $(a + (a-1)b)$ number of vertices is

$$ABC(W_{a,b}) = a_i b_j\sqrt{\frac{b+2}{(b+3)}} + a_i\frac{\sqrt{2(b+2)}}{(b+3)} + a_i\sqrt{\frac{a+b}{(a-1)(b+3)}}$$

Proof: Here, we independently examine the pair of vertices of W_a between the pair of vertices u_i for $i = 1,2,3, \ldots, b$ of $W_{a,b}$ along with the pair of vertices that belongs to W_a and otherwise degree 1 vertex. Let $v_1, v_2, v_3, \ldots., v_a$ be the vertices of wheel graph W_a where $i\ ranges\ from$ 1,2,3,4, …, a and let $u_1, u_2, u_3, \ldots, u_b$ be a degree one vertices of $W_{a,b}$ where $j = 1,2,3, \ldots., b$. (As in Figure. 6). Let $d(v_i) = (b+3)\ for\ i = 2,3,4, \ldots. a, d(v_1) = (a-1)$ and $d(u_j) = 1$. Then we have,

$$ABC(W_{a,b}) = \sum_{i=2,j=1}^{a_i,b_j} \sqrt{\frac{d(v_i)+d(u_j)-2}{d(v_i)d(u_j)}} + \sum_{i=2}^{a_i} \sqrt{\frac{d(v_i)+d(v_i)-2}{d(v_i)d(v_i)}} + \sum_{i=2}^{a_i} \sqrt{\frac{d(v_1)+d(v_i)-2}{d(v_1)d(v_i)}}$$

$$= \sum_{i=2,j=1}^{a_i,b_j} \sqrt{\frac{(b+3)+1-2}{(1)(b+3)}} + \sum_{i=2}^{a_i} \sqrt{\frac{(b+3)+(b+3)-2}{(b+3)^2}} + \sum_{i=2}^{a_i} \sqrt{\frac{(a-1)+(b+3)-2}{(a-1)(b+3)}}$$

$$= a_i b_j\sqrt{\frac{b+2}{(b+3)}} + a_i\sqrt{\frac{2(b+2)}{(b+3)^2}} + a_i\sqrt{\frac{a+b}{(a-1)(b+3)}}$$

$$= a_i b_j\sqrt{\frac{b+2}{(b+3)}} + a_i\frac{\sqrt{2(b+2)}}{(b+3)} + a_i\sqrt{\frac{a+b}{(a-1)(b+3)}}$$

Theorem 3.16: The geometric-arithmetic index of a thorn wheel graph having $(a + (a-1)b)$ number of vertices is

$$GA(W_{a,b}) = a_i b_j \frac{2\sqrt{(b+3)}}{(b+4)} + a_i + a_i \frac{2\sqrt{(a-1)(b+3)}}{(a+b+2)}.$$

Proof: Here, we independently examine the pair of vertices of W_a between the pair of vertices u_i, for $i = 1,2,3,\ldots,b$ of $W_{a,b}$ along with the pair of vertices that belongs to W_a and otherwise degree 1 vertex. Let $v_1, v_2, v_3, \ldots, v_a$ be the vertices of wheel graph W_a where $i\ ranges\ from$ $1,2,3,4,\ldots,a$ and let $u_1, u_2, u_3, \ldots, u_b$ be a degree one vertices of $W_{a,b}$ where $j = 1,2,3,\ldots,b$. (As in Figure. 6). Let $d(v_i) = (b+3)\ for\ i = 2,3,4,\ldots.a, d(v_1) = (a-1)$ and $d(u_j) = 1$. Then we have,

$$GA(W_{a,b}) = \sum_{i=2,j=1}^{a_i,b_j} \frac{2\sqrt{d(v_i)d(u_j)}}{(d(v_i)+d(u_j))} + \sum_{i=2}^{a_i} \frac{2\sqrt{d(v_i)d(v_i)}}{(d(v_i)+d(v_i))} + \sum_{i=2}^{a_i} \frac{2\sqrt{d(v_1)d(v_i)}}{(d(v_1)+d(v_i))}$$

$$= \sum_{i=2,j=1}^{a_i,b_j} \frac{2\sqrt{1(b+3)}}{(b+3+1)} + \sum_{i=2}^{a_i} \frac{2\sqrt{(b+3)(b+3)}}{((b+3)+(b+3))} + \sum_{i=2}^{a_i} \frac{2\sqrt{(a-1)(b+3)}}{((a-1)+(b+3))}$$

$$= a_i b_j \frac{2\sqrt{(b+3)}}{(b+4)} + a_i \frac{2\sqrt{(b+3)^2}}{2(b+3)} + a_i \frac{2\sqrt{(a-1)(b+3)}}{(a+b+2)}$$

$$= a_i b_j \frac{2\sqrt{(b+3)}}{(b+4)} + a_i + a_i \frac{2\sqrt{(a-1)(b+3)}}{(a+b+2)}.$$

3.1.8. A Cog-Wheel graph ${W^c}_a$.

${W^c}_a$ is the graph formed from a star graph $W_a\ (a \geq 4)$ of the vertex set $\{v_1, v_2, v_3, \ldots, v_{a-1}, v_a\}$ with the addition of $(b-1)$ number of vertices such as $\{u_1, u_2, u_3, \ldots, u_{b-1}\}$ and edges given by $\{u_j v_{i+1}, u_j v_{i+2}: i = 1,2,3,\ldots,a\ and\ j = 1,2,3,\ldots,(b-1)\}(v_{a+1} = v_2)$, as shown in Figure 7.
We know that $p({W^c}_a) = (a+b-1), q({W^c}_a) = 2(a+b-2)$.

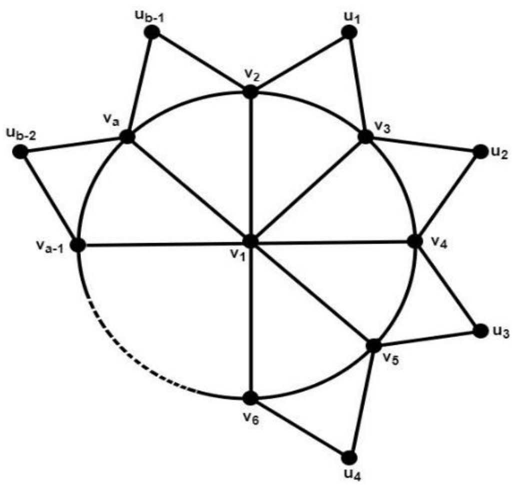

Figure 7. A Cog-Wheel graph ${W^c}_a$.

3.1.9. The thorn cog-wheel graph ${W^{c^*}}_a$.

${W^{c^*}}_a$ is the graph formed from a $W_a^c\ (a \geq 4)$, of the vertex set $\{v_1, v_2, v_3, \ldots, v_{a-1}, v_a, u_1, u_2, u_3, \ldots, u_{b-1}\}$ for $i = 1,2,3,\ldots,a\ and\ j = 1,2,3,\ldots,(b-$

1) with the addition of $2(b-1)$ number of vertices such as $\{w_1, w_2, w_3, \ldots\ldots, w_{2b-3}, w_{2b-2}\}$ and edges given by $\{u_j w_{2j-1}, u_j w_{2j}: j = 1,2,3, \ldots\ldots, (b-1)\}$, as shown in Figure 8.

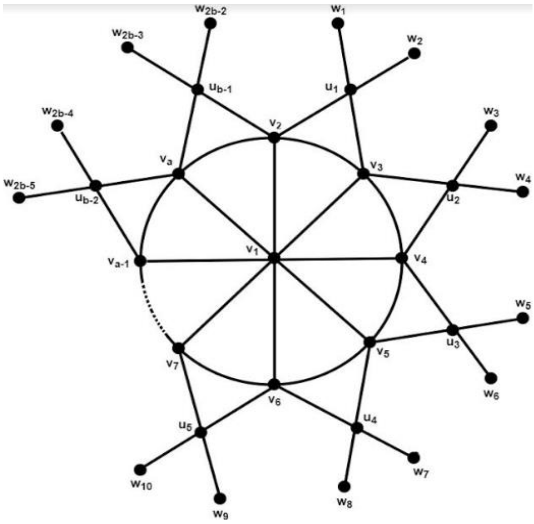

Figure 8. The Thorn Cog-Wheel graph $W^{c^*}{}_a$.

Theorem 3.17: The reciprocal randic index of a thorn cog-wheel graph having $(a + 3(b-1))$ number of vertices is

$$R^{-1}(W^{c^*}{}_a) = 2a_i(b_j - 1)\sqrt{5} + a_i\sqrt{5(a-1)} + a_i 5 + 4(b_j - 1)(b_k - 1).$$

Proof: Here we independently examine the pair of vertices of W_a between the pair of vertices u_j for $j = 1,2,3, \ldots, (b-1)$ of $W^c{}_a$ along with the pair of vertices that belongs to W_a and between the pair of vertices u_j for $j = 1,2,3, \ldots, (b-1)$ of $W^c{}_a$ along with the pair of vertices degree 1 vertices of $W^{c^*}{}_a$. Let $v_1, v_2, v_3, \ldots\ldots, v_a$ be the vertices of wheel graph W_a, where $i\ ranges\ from$ $1,2,3,4, \ldots, a$ and let $u_1, u_2, u_3, \ldots, u_{b-1}$ be addition of $(b-1)$ number of vertices of $W^c{}_a$ where $j = 1,2,3, \ldots\ldots, (b-1)$. Also, let $w_1, w_2, \ldots\ldots, w_{2b-2}$, $k = 1,2,3, \ldots\ldots, (2b-2)$ of $W^{c^*}{}_a$ be an additional $(2b-2)$ number of degree 1 vertices of $W^{c^*}{}_a$. (As in Figure. 8). Let $d(v_1) = (a-1)$, $d(u_j) = 4(for\ j = 1,2, \ldots, (b-1))$, $d(v_i) = 5(for\ i = 2,3,4, \ldots\ldots, a)$ and $d(w_k) = 1(for\ k = 1,2,3, \ldots\ldots, 2(b-1))$. Then we have,

$$R^{-1}(W^{c^*}{}_a) = \sum_{i=2,j=1}^{a_i,(b_j-1)} \sqrt{(d(v_i))(d(u_j))} + \sum_{i=2}^{a_i} \sqrt{(d(v_1))(d(v_i))} + \sum_{i=2}^{a_i} \sqrt{(d(v_i))^2} + \sum_{j,k=1}^{b_j-1,2b_k-2} \sqrt{(d(u_j))(d(w_k))}$$

$$= \sum_{i=2,j=1}^{a_i,(b_j-1)} \sqrt{5(4)} + \sum_{i=2}^{a_i} \sqrt{5(a-1)} + \sum_{i=2}^{a_i} \sqrt{(5)^2} + \sum_{j,k=1}^{b_j-1,2b_k-2} \sqrt{4(1)}$$

$$= a_i(b_j - 1)2\sqrt{5} + a_i\sqrt{(a-1)5} + a_i 5 + (b_j - 1)(2b_k - 2)2$$

$$= 2a_i(b_j - 1)\sqrt{5} + a_i\sqrt{5(a-1)} + a_i 5 + 4(b_j - 1)(b_k - 1).$$

Theorem 3.18: The reduced reciprocal randic index of a thorn cog-wheel graph having $(a+3(b-1))$ number of vertices is

$$RR^{-1}(W^{c^*}{}_a) = 2a_i(b_j-1)\sqrt{3} + 2a_i\sqrt{(a-2)} + 4a_i.$$

Proof: Here we independently examine the pair of vertices of W_a between the pair of vertices u_j for $j = 1,2,3,\dots,(b-1)$ of $W^c{}_a$ along with the pair of vertices that belongs to W_a and between the pair of vertices u_j for $j = 1,2,3,\dots,(b-1)$ of $W^c{}_a$ along with the pair of vertices degree 1 vertices of $W^{c^*}{}_a$. Let $v_1, v_2, v_3, \dots., v_a$ be the vertices of wheel graph W_a where $i\ ranges\ from$ $1,2,3,4,\dots,a$ and let $u_1, u_2, u_3, \dots, u_{b-1}$ be addition of $(b-1)$ number of vertices of $W^c{}_a$ where $j = 1,2,3,\dots.,(b-1)$. Also, let $w_1, w_2, \dots\dots, w_{2b-2}$, $k = 1,2,3,\dots.,(2b-2)$ of $W^{c^*}{}_a$ be an additional $(2b-2)$ number of degree 1 vertices of $W^{c^*}{}_a$. (As in Figure. 8). Let $d(v_1) = (a-1)$, $d(u_j) = 4(for\ j = 1,2,\dots,(b-1))$, $d(v_i) = 5(for\ i = 2,3,4,\dots..,a)$ and $d(w_k) = 1(for\ k = 1,2,3,\dots\dots,2(b-1))$. Then we have,

$$RR^{-1}(W^{c^*}{}_a) = \sum_{i=2,j=1}^{a_i,(b_j-1)} \sqrt{(d(v_i)-1)(d(u_j)-1)} + \sum_{i=2}^{a_i} \sqrt{(d(v_1)-1)(d(v_i)-1)}$$

$$+ \sum_{i=2}^{a_i} \sqrt{(d(v_i)-1)^2} + \sum_{j,k=1}^{b_j-1,2b_k-2} \sqrt{(d(u_j)-1)(d(w_k)-1)}$$

$$= \sum_{i=2,j=1}^{a_i,(b_j-1)} \sqrt{(5-1)(4-1)} + \sum_{i=2}^{a_i} \sqrt{(a-2)4}$$

$$+\sum_{i=2}^{a_i} \sqrt{(4)^2} + \sum_{j,k=1}^{b_j-1,2b_k-2} \sqrt{3(0)}$$

$$= 2a_i(b_j-1)\sqrt{3} + 2a_i\sqrt{(a-2)} + 4a_i.$$

Theorem 3.19: The Atom-bond connectivity index of a thorn cog-wheel graph having $(a+3(b-1))$ number of vertices is

$$ABC(W^{c^*}{}_a) = a_i(b_j-1)\frac{\sqrt{7}}{2\sqrt{5}} + a_i\sqrt{\frac{a+2}{5(a-1)}} + a_i\frac{2\sqrt{2}}{5} + (b_j-1)(b_k-1)\sqrt{3}.$$

Proof: Here we independently examine the pair of vertices of W_a between the pair of vertices u_j for $j = 1,2,3,\dots,(b-1)$ of $W^c{}_a$ along with the pair of vertices that belongs to W_a and between the pair of vertices u_j for $j = 1,2,3,\dots,(b-1)$ of $W^c{}_a$ along with the pair of vertices degree 1 vertices of $W^{c^*}{}_a$. Let $v_1, v_2, v_3, \dots., v_a$ be the vertices of wheel graph W_a where $i\ ranges\ from$ $1,2,3,4,\dots,a$ and let $u_1, u_2, u_3, \dots, u_{b-1}$ be addition of $(b-1)$ number of vertices of $W^c{}_a$ where $j = 1,2,3,\dots.,(b-1)$. Also, let $w_1, w_2, \dots\dots, w_{2b-2}$, $k = 1,2,3,\dots.,(2b-2)$ of $W^{c^*}{}_a$ be an additional $(2b-2)$ number of degree 1 vertices of $W^{c^*}{}_a$. (As in Figure. 8). Let $d(v_1) = (a-1)$, $d(u_j) = 4(for\ j = 1,2,\dots,(b-1))$, $d(v_i) = 5(for\ i = 2,3,4,\dots..,a)$ and $d(w_k) = 1(for\ k = 1,2,3,\dots\dots,2(b-1))$. Then we have,

$$ABC(W^{c^*}{}_a) = \sum_{i=2,j=1}^{a_i,(b_j-1)} \sqrt{\frac{d(v_i)+d(u_j)-2}{d(v_i)d(u_j)}} + \sum_{i=2}^{a_i} \sqrt{\frac{d(v_1)+d(v_i)-2}{d(v_1)d(v_i)}} + \sum_{i=2}^{a_i} \sqrt{\frac{d(v_i)+d(v_i)-2}{d(v_i)d(v_i)}} + \sum_{j,k=1}^{b_j-1,2b_k-2} \sqrt{\frac{d(u_j)+d(w_k)-2}{d(u_j)d(w_k)}}$$

$$= \sum_{i=2,j=1}^{a_i,(b_j-1)} \sqrt{\frac{5+4-2}{5(4)}} + \sum_{i=2}^{a_i} \sqrt{\frac{(a-1)+(5)-2}{5(a-1)}} + \sum_{i=2}^{a_i} \sqrt{\frac{(5)+(5)-2}{5(5)}} + \sum_{j,k=1}^{b_j-1,2b_k-2} \sqrt{\frac{4+1-2}{4(1)}}$$

$$= a_i(b_j-1)\sqrt{\frac{7}{4(5)}} + a_i\sqrt{\frac{a+2}{5(a-1)}} + a_i\frac{2\sqrt{2}}{5} + (b_j-1)(2b_k-2)\frac{\sqrt{3}}{2}$$

$$= a_i(b_j-1)\frac{\sqrt{7}}{2\sqrt{5}} + a_i\sqrt{\frac{a+2}{5(a-1)}} + a_i\frac{2\sqrt{2}}{5} + (b_j-1)(b_k-1)\sqrt{3}.$$

Theorem 3.20: The geometric-arithmetic index of a thorn cog-wheel graph having $(a+3(b-1))$ number of vertices is

$$GA(W^{c^*}{}_a) = a_i(b_j-1)\frac{4\sqrt{5}}{9} + a_i\frac{2\sqrt{5(a-1)}}{(a+4)} + a_i + \frac{8(b_j-1)(b_k-1)}{5}.$$

Proof: Here, we independently examine the pair of vertices of W_a between the pair of vertices u_j for $j = 1,2,3,\dots,(b-1)$ of $W^c{}_a$ along with the pair of vertices that belongs to W_a and between the pair of vertices u_j for $j = 1,2,3,\dots,(b-1)$ of $W^c{}_a$ along with the pair of vertices degree 1 vertices of $W^{c^*}{}_a$. Let $v_1, v_2, v_3, \dots, v_a$ be the vertices of wheel graph W_a where i *ranges from* $1,2,3,4,\dots,a$ and let $u_1, u_2, u_3, \dots, u_{b-1}$ be addition of $(b-1)$ number of vertices of $W^c{}_a$ where $j = 1,2,3,\dots,(b-1)$. Also, let $w_1, w_2, \dots, w_{2b-2}$, $k = 1,2,3,\dots,(2b-2)$ *of* $W^{c^*}{}_a$ be an additional $(2b-2)$ number of degree 1 vertices of $W^{c^*}{}_a$. (As in Figure. 8). Let $d(v_1) = (a-1)$, $d(u_j) = 4(for\ j = 1,2,\dots,(b-1))$, $d(v_i) = 5(for\ i = 2,3,4,\dots,a)$ and $d(w_k) = 1(for\ k = 1,2,3,\dots,2(b-1))$. Then we have,

$$GA(W^{c^*}{}_a) = \sum_{i=2,j=1}^{a_i,(b_j-1)} \frac{2\sqrt{d(v_i)d(u_j)}}{d(v_i)+d(u_j)} + \sum_{i=2}^{a_i} \frac{2\sqrt{d(v_1)d(v_i)}}{d(v_1)+d(v_i)} + \sum_{i=2}^{a_i} \frac{2\sqrt{(d(v_i))^2}}{d(v_i)+d(v_i)} + \sum_{j,k=1}^{b_j-1,2b_k-2} \frac{2\sqrt{d(u_j)d(w_k)}}{d(u_j)+d(w_k)}$$

$$= \sum_{i=2,j=1}^{a_i,(b_j-1)} \frac{2\sqrt{5(4)}}{(5+4)} + \sum_{i=2}^{a_i} \frac{2\sqrt{5(a-1)}}{(a-1+5)} + \sum_{i=2}^{a_i} \frac{2\sqrt{(5)^2}}{10} + \sum_{j,k=1}^{b_j-1,2b_k-2} \frac{2\sqrt{4(1)}}{5}$$

$$= a_i(b_j-1)\frac{4\sqrt{5}}{9} + a_i\frac{2\sqrt{5(a-1)}}{(a+4)} + a_i + (b_j-1)(2b_k-2)\frac{4}{5}$$

$$= a_i(b_j - 1)\frac{4\sqrt{5}}{9} + a_i\frac{2\sqrt{5(a-1)}}{(a+4)} + a_i + \frac{8(b_j-1)(b_k-1)}{5}.$$

4. Conclusions

In this paper, we have computed reciprocal randic(R^{-1}) index, reduced reciprocal randic$\left(RR^{-1}\right)$ index, atom-bond connectivity(ABC) index, geometric-arithmetic(GA) index of complete thorn graph and wheel graph along with its special cases. These descriptors constitute the biological and physicochemical properties of chemical compounds. Topological indices have many applications in the field of modern chemistry.

Funding

This research received no external funding.

Acknowledgments

We are grateful to our institutes for their support.

Conflicts of Interest

The authors declare no conflict of interest.

References

1. Gutman, I. Distance in thorny graph. *Publ. Inst. Math. Beograd* **1998**, 63, 31-36.
2. Knor, M.; Potcnik, P.; Skrekovski, R. Relationship between the edge-Winer index and the Gutman index of a graph. *Discrete Appl. Math.* **2014**, *167*, 197-201, https://doi.org/10.1016/j.dam.2013.12.009.
3. Vukicevic, D.; Graovac, A. On modified Wiener indices of thorn graphs. *MATCH - Communications in Mathematical and in Computer Chemistry* **2004**, *50*, 93-108.
4. Vukicevic, D.; Nikolić, S.; Trinajstić, N. On the Schultz Index of Thorn Graphs. *Internet Electronic Journal of Molecular Design* **2005**, *4*, 501-514.
5. Zhou, B. On modified Wiener indices of thorn trees. *Kragujevac J. Math.* **2005**, *27*, 5-9.
6. Wu, B. Wiener index of line graphs. *MATCH Commun. Math. Comput. Chem.* **2010**, *64*, 699–706.
7. Wei, J.; Imran, M.; Iqbal, M.A.; Zaighum, M.A. Zagreb-Type Indices of-Vertex Join and-Edge Join of Graphs. *Journal of Chemistry* **2020**, *2020*, https://doi.org/10.1155/2020/9767128.
8. Li, Q.Z.; Nazar, K.; Ahmed, I.; Tlili, I. Valency-Based Descriptors for Silicon Carbides, Bismuth (III) Iodide, and Dendrimers in Drug Applications. *Journal of Chemistry* **2020**, *2020*, https://doi.org/10.1155/2020/8616309.
9. Pio, J.; Thayamathy, Panchadcharam, E.; Masilamani, K. Computing Zagreb and Randic Indices of PEG-Cored Dendrimers Used for Drug and Gene Delivery. *Journal of Chemistry* **2020**, *2020*, https://doi.org/10.1155/2020/2821510.
10. Huang, Y.; Mutee-ur-Rehman, H.; Nazeer, S.; Afzal, D.; Qiang, X. Some Bounds of Weighted Entropies with Augmented Zagreb Index Edge Weights. *Discrete Dynamics in Nature and Society* **2020**, *2020*, https://doi.org/10.1155/2020/3562382.
11. Kwun, Y.C.; Rehman, H.M.u.; Yousaf, M.; Nazeer, W.; Kang, S.M. The Entropy of Weighted Graphs with Atomic Bond Connectivity Edge Weights. *Discrete Dynamics in Nature and Society* **2018**, *2018*, https://doi.org/10.1155/2018/8407032.
12. Li, X.; Ahmad, M.; Javaid, M.; Saeed, M.; Liu, J.-B. Bounds on General Randic Index for F-Sum Graphs. *Journal of Mathematics* **2020**, *2020*, https://doi.org/10.1155/2020/9129365.
13. Imran, M.; Akhter, S.; Iqbal, Z. On the Eccentric Connectivity Polynomial of $\mathscr{F}$-Sum of Connected Graphs. *Complexity* **2020**, 2020, https://doi.org/10.1155/2020/5061682.
14. Tang, J.-H.; Ali, U.; Javaid, M.; Shabbir, K. Zagreb Connection Indices of Subdivision and Semi-Total Point Operations on Graphs. *Journal of Chemistry* **2019**, *2019*, https://doi.org/10.1155/2019/9846913.